Replica method for the p-spin model

PHYS-642

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Contents

1 Setting

The Hamiltonian for the p-spin model is

$$\mathcal{H}(\mathbf{S}) = -\sum_{i_1 < \cdots < i_p} J_{i_1,\dots,i_p} S_{i_1} \cdots S_{i_p}$$

where the interaction terms follow a Gaussian distribution with standard deviation $\sigma = \sqrt{\frac{p!}{2N^{p-1}}}$:

$$\mathbb{P}(J) = \sqrt{\frac{N^{p-1}}{\pi p!}} \exp\left(-\frac{N^{p-1}}{p!} J^2\right)$$

Remember the replica trick: as $n \to 0$,

$$Z^n = \mathrm{e}^{n \log Z} = 1 + n \log Z + o(n)$$

which gives us

$$\mathbb{E}\left[\log Z\right] = \lim_{n \to 0} \frac{\mathbb{E}\left[Z^n\right] - 1}{n}$$

This explains why we want to compute powers of Z, and their expectation with respect to the disorder $\mathbf{J}$. This will in turn allow us to deduce the free entropy density

$$\Phi = \lim_{N \to \infty} \frac{\mathbb{E}\left[\log Z\right]}{N} \qquad \text{so} \qquad \Phi_{\text{RS}} = \lim_{N \to \infty} \lim_{n \to 0} \frac{\mathbb{E}\left[Z^n\right] - 1}{nN} \tag{1}$$

2 Powers of the partition function

$$
\begin{aligned}
Z^n &= \left(\sum_{\mathbf{S} \in \{\pm 1\}^N} \exp\left[-\beta \mathcal{H}(\mathbf{S})\right] \right)^n = \sum_{\mathbf{S}^1, \ldots, \mathbf{S}^n} \exp\left[-\beta \sum_{a=1}^n \mathcal{H}(\mathbf{S}^a) \right] \\
&= \sum_{\mathbf{S}^1, \ldots, \mathbf{S}^n} \exp\left[\beta \sum_{i_1 < \cdots < i_p} J_{i_1, \ldots, i_p} \sum_{a=1}^n S_{i_1}^a \cdots S_{i_p}^a \right] \\
&= \sum_{\mathbf{S}^1, \ldots, \mathbf{S}^n} \prod_{i_1 < \cdots < i_p} \exp\left[\beta J_{i_1, \ldots, i_p} \sum_{a=1}^n S_{i_1}^a \cdots S_{i_p}^a \right]
\end{aligned}
$$

We take the expectation with respect to the disorder, using the independence of the $J_{i_1, \ldots, i_p}$:

$$
\mathbb{E}[Z^n] = \sum_{\mathbf{S}^1, \ldots, \mathbf{S}^n} \prod_{i_1 < \cdots < i_p} \mathbb{E} \exp\left[\beta J_{i_1, \ldots, i_p} \sum_{a=1}^n S_{i_1}^a \cdots S_{i_p}^a \right]
$$

Now is a good time to remember that if $X \sim \mathcal{N}(0, 1)$, then

$$
\mathbb{E}_X[e^{\sigma X}] = \mathrm{e}^{\sigma^2/2} \tag{2}
$$

We apply the previous result with $\sigma = \beta \sqrt{\frac{p!}{2N^{p-1}}} \sum_{a=1}^n S_{i_1}^a \cdots S_{i_p}^a$:

$$
\begin{aligned}
\mathbb{E}[Z^n] &= \sum_{\mathbf{S}^1, \ldots, \mathbf{S}^n} \prod_{i_1 < \cdots < i_p} \exp\left[\frac{1}{2} \beta^2 \frac{p!}{2N^{p-1}} \left(\sum_{a=1}^n S_{i_1}^a \cdots S_{i_p}^a \right)^2 \right] \\
&= \sum_{\mathbf{S}^1, \ldots, \mathbf{S}^n} \exp\left[\frac{\beta^2 p!}{4N^{p-1}} \sum_{i_1 < \cdots < i_p} \left(\sum_{a=1}^n S_{i_1}^a \cdots S_{i_p}^a \right)^2 \right] \\
&= \sum_{\mathbf{S}^1, \ldots, \mathbf{S}^n} \exp\left[\frac{\beta^2}{4N^{p-1}} \sum_{i_1, \ldots, i_p} \sum_{a,b=1}^n S_{i_1}^a \cdots S_{i_p}^a S_{i_1}^b \cdots S_{i_p}^b \right] \\
&= \sum_{\mathbf{S}^1, \ldots, \mathbf{S}^n} \exp\left[\frac{\beta^2}{4N^{p-1}} \sum_{i_1, \ldots, i_p} \sum_{a,b=1}^n (S_{i_1}^a S_{i_1}^b) \cdots (S_{i_p}^a S_{i_p}^b) \right] \\
&= \sum_{\mathbf{S}^1, \ldots, \mathbf{S}^n} \exp\left[\frac{\beta^2}{4N^{p-1}} \sum_{a,b=1}^n \left(\sum_{i=1}^N S_i^a S_i^b \right)^p \right]
\end{aligned}
$$

We can thus conclude:

$$
\mathbb{E}[Z^n] = \sum_{\mathbf{S}^1, \ldots, \mathbf{S}^n} \exp\left[\frac{N\beta^2}{4} \sum_{a,b=1}^n \left(\frac{\mathbf{S}^a \cdot \mathbf{S}^b}{N} \right)^p \right] \tag{3}
$$

where $\mathbf{S}^a \cdot \mathbf{S}^b$ denotes an inner product.

3 Reminders on the Dirac distribution

For the replica trick, we need use the Dirac distribution δ, defined on any test function φ by $\langle \delta, \varphi \rangle = \varphi(0)$. It can also be written

$$\int_{\mathbb{R}} \mathrm{d}x\, \delta(x)\varphi(x) = \varphi(0)$$

What does a change of variable mean "inside" the Dirac distribution? We can deduce it from its action on the test function:

$$\int_{\mathbb{R}} \mathrm{d}x\, \delta(\gamma x - \mu)\varphi(x) = \int_{\mathbb{R}} \frac{\mathrm{d}y}{\gamma}\, \delta(y)\varphi\left(\frac{y+\mu}{\gamma}\right) = \frac{1}{\gamma}\varphi\left(\frac{\mu}{\gamma}\right) \tag{4}$$

We also recall the following Fourier transform:

$$\delta(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\omega x} \tag{5}$$

4 Overlap formulation

We can work some more on Equation (3) by noting that the roles of a and b are symmetrical, except for the simple case $a = b$ where the replicas are aligned:

$$\begin{aligned} \mathbb{E}[Z^n] &= \sum_{\mathbf{S}^1,\ldots,\mathbf{S}^n} \exp\left[\frac{N\beta^2}{4}\left(n + 2\sum_{a<b}\left(\frac{\mathbf{S}^a\cdot\mathbf{S}^b}{N}\right)^p\right)\right] \\ &= \sum_{\mathbf{S}^1,\ldots,\mathbf{S}^n} \exp\left[\frac{N\beta^2 n}{4}\right]\prod_{a<b}\exp\left[\frac{N\beta^2}{2}\left(\frac{\mathbf{S}^a\cdot\mathbf{S}^b}{N}\right)^p\right] \end{aligned} \tag{6}$$

We define the overlap Q^{ab} between two configurations $\mathbf{S}^a$ and $\mathbf{S}^b$ as follows:

$$Q^{ab} = \frac{\mathbf{S}^a\cdot\mathbf{S}^b}{N}$$

And we want to use Equation (4) on the Dirac distribution, with

$$\varphi(x) = \exp\left[\frac{N\beta^2}{2}x^p\right] \qquad \mu = \mathbf{S}^a\cdot\mathbf{S}^b \qquad \gamma = N$$

On the one hand,

$$\frac{1}{\gamma}\varphi\left(\frac{\mu}{\gamma}\right) = \frac{1}{N}\exp\left[\frac{N\beta^2}{2}\left(\frac{\mathbf{S}^a\cdot\mathbf{S}^b}{N}\right)^p\right]$$

On the other hand,

$$\int_{\mathbb{R}} \mathrm{d}x\, \delta\left(\gamma x - \mu\right)\varphi(x) = \int_{\mathbb{R}} \mathrm{d}Q^{ab}\, \exp\left[\frac{N\beta^2}{2}\left(Q^{ab}\right)^p\right]\delta\left(NQ^{ab} - \mathbf{S}^a\cdot\mathbf{S}^b\right)$$

We plug this back into Equation (6):

$$\mathbb{E}[Z^n] = \sum_{\mathbf{S}^1,\ldots,\mathbf{S}^n} \exp\left[\frac{N\beta^2 n}{4}\right]\prod_{a<b} N\int_{\mathbb{R}} \mathrm{d}Q^{ab}\, \exp\left[\frac{N\beta^2}{2}\left(Q^{ab}\right)^p\right]\delta\left(NQ^{ab} - \mathbf{S}^a\cdot\mathbf{S}^b\right)$$

Applying the Fourier transform of Equation (5) on top of this, we get:

$$\begin{aligned}
\mathbb{E}[Z^n] &= \sum_{\mathbf{S}^1,\dots,\mathbf{S}^n} \exp\left[\frac{N\beta^2 n}{4}\right] \prod_{a<b} N \int_{\mathbb{R}} \mathrm{d}Q^{ab} \, \exp\left[\frac{N\beta^2}{2}\left(Q^{ab}\right)^p\right] \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}\widehat{Q}^{ab} \exp\left[\mathrm{i}\widehat{Q}^{ab}\left(NQ^{ab} - \mathbf{S}^a \cdot \mathbf{S}^b\right)\right] \\
&= \sum_{\mathbf{S}^1,\dots,\mathbf{S}^n} \left(\frac{N}{2\pi}\right)^{\frac{n(n-1)}{2}} \iint \mathrm{d}\mathbf{Q}\, \mathrm{d}\widehat{\mathbf{Q}} \, \exp\left[\frac{N\beta^2 n}{4} + \frac{N\beta^2}{2}\sum_{a<b}\left(Q^{ab}\right)^p + \mathrm{i}N\sum_{a<b}\widehat{Q}^{ab}Q^{ab} - \mathrm{i}\sum_{a<b}\widehat{Q}^{ab}\left(\mathbf{S}^a\cdot\mathbf{S}^b\right)\right] \\
&= \left(\frac{N}{2\pi}\right)^{\frac{n(n-1)}{2}} \iint \mathrm{d}\mathbf{Q}\, \mathrm{d}\widehat{\mathbf{Q}} \, \exp\left[N\left(\frac{\beta^2 n}{4} + \frac{\beta^2}{2}\sum_{a<b}\left(Q^{ab}\right)^p + \mathrm{i}\sum_{a<b}\widehat{Q}^{ab}Q^{ab}\right)\right] \sum_{\mathbf{S}^1,\dots,\mathbf{S}^n} \exp\left[-\mathrm{i}\sum_{a<b}\widehat{Q}^{ab}\left(\mathbf{S}^a\cdot\mathbf{S}^b\right)\right]
\end{aligned}$$

5 Saddle point

We rewrite the last factor as

$$\begin{aligned}
\sum_{\mathbf{S}^1,\dots,\mathbf{S}^n} \exp\left[-\mathrm{i}\sum_{a<b}\widehat{Q}^{ab}\left(\mathbf{S}^a\cdot\mathbf{S}^b\right)\right] &= \exp\left[N\frac{1}{N}\log\sum_{\mathbf{S}^1,\dots,\mathbf{S}^n}\exp\left[-\mathrm{i}\sum_{a<b}\widehat{Q}^{ab}\left(\mathbf{S}^a\cdot\mathbf{S}^b\right)\right]\right] \\
&= \exp\left[N\frac{1}{N}\log\sum_{\mathbf{S}^1,\dots,\mathbf{S}^n}\exp\left[-\mathrm{i}\sum_{a<b}\widehat{Q}^{ab}\sum_{i=1}^{N}S_i^aS_i^b\right]\right] \\
&= \exp\left[N\frac{1}{N}\log\sum_{\mathbf{S}^1,\dots,\mathbf{S}^n}\prod_{i=1}^{N}\exp\left[-\mathrm{i}\sum_{a<b}\widehat{Q}^{ab}S_i^aS_i^b\right]\right] \\
&= \exp\left[N\frac{1}{N}\log\prod_{i=1}^{N}\sum_{S_i^1,\dots,S_i^n}\exp\left[-\mathrm{i}\sum_{a<b}\widehat{Q}^{ab}S_i^aS_i^b\right]\right] \\
&= \exp\left[N\frac{1}{N}\log\left(\sum_{S^1,\dots,S^n}\exp\left[-\mathrm{i}\sum_{a<b}\widehat{Q}^{ab}S^aS^b\right]\right)^N\right] \\
&= \exp\left[N\log\sum_{S^1,\dots,S^n}\exp\left[-\mathrm{i}\sum_{a<b}\widehat{Q}^{ab}S^aS^b\right]\right]
\end{aligned}$$

We can now conclude:

$$\mathbb{E}[Z^n] = \left(\frac{N}{2\pi}\right)^{\frac{n(n-1)}{2}} \iint \mathrm{d}\mathbf{Q}\, \mathrm{d}\widehat{\mathbf{Q}} \, \exp\left[-NG^{(n)}(\mathbf{Q},\widehat{\mathbf{Q}})\right]$$

where the function $G^{(n)}(\mathbf{Q},\widehat{\mathbf{Q}})$ is given by

$$G^{(n)}(\mathbf{Q},\widehat{\mathbf{Q}}) = -\left(\frac{\beta^2 n}{4} + \frac{\beta^2}{2}\sum_{a<b}\left(Q^{ab}\right)^p + \mathrm{i}\sum_{a<b}\widehat{Q}^{ab}Q^{ab}\right) - \log\sum_{S^1,\dots,S^n}\exp\left[-\mathrm{i}\sum_{a<b}\widehat{Q}^{ab}S^aS^b\right] \tag{7}$$

In the thermodynamic limit, we can apply the saddle point (or Laplace) approximation to get

$$\mathbb{E}[Z^n] \asymp \exp\left[-NG^{(n)}(\mathbf{Q}_\star^{(n)},\widehat{\mathbf{Q}}_\star^{(n)})\right] \tag{8}$$

where the pair $(\mathbf{Q}_\star^{(n)},\widehat{\mathbf{Q}}_\star^{(n)})$ extremizes $G^{(n)}$ over the complex plane!

6 Replica symmetry

To find extremas of $G^{(n)}$, we make a replica symmetry Ansatz, and look for solutions of the form

$$\forall a < b, \qquad Q^{ab} = Q \qquad \text{and} \qquad \widehat{Q}^{ab} = \mathrm{i}\widehat{Q}$$

This greatly simplifies Equation (7):

$$G^{(n)}(Q, \widehat{Q}) = -\frac{\beta^2 n}{4} - \frac{\beta^2}{2}\frac{n(n-1)}{2}Q^p + \frac{n(n-1)}{2}\widehat{Q}Q - \log \sum_{S^1,\dots,S^n} \exp\left[\widehat{Q}\sum_{a<b} S^a S^b\right]$$

Now we need to decouple replicas in the last term by expressing it as $\mathrm{e}^{\sigma^2/2}$. So we reinsert all the terms $a \geq b$ in the sum:

$$\begin{aligned}
\exp\left[\widehat{Q}\sum_{a<b} S^a S^b\right] &= \exp\left[\frac{1}{2}\left(\widehat{Q}\sum_{a,b} S^a S^b - \widehat{Q}\sum_{a=b} S^a S^b\right)\right] \\
&= \exp\left[\frac{1}{2}\left(\sqrt{\widehat{Q}}\sum_a S^a\right)^2\right] \exp\left[-\frac{n}{2}\widehat{Q}\right]
\end{aligned}$$

Now we apply Equation (2) with $c = 1$ and $\sigma = \sqrt{\widehat{Q}}\sum_a S^a$: if $X \sim \mathcal{N}(0,1)$, then

$$\begin{aligned}
\exp\left[\widehat{Q}\sum_{a<b} S^a S^b\right] &= \exp\left[-\frac{n}{2}\widehat{Q}\right] \mathbb{E}_X \exp\left[\left(\sqrt{\widehat{Q}}\sum_a S^a\right) X\right] \\
&= \exp\left[-\frac{n}{2}\widehat{Q}\right] \mathbb{E}_X \prod_a \exp\left[\sqrt{\widehat{Q}} S^a X\right]
\end{aligned}$$

We can now simplify the log term:

$$\begin{aligned}
\log \sum_{S^1,\dots,S^n} \exp\left[\widehat{Q}\sum_{a<b} S^a S^b\right] &= -\frac{n}{2}\widehat{Q} + \log \sum_{S^1,\dots,S^n} \mathbb{E}_X \prod_a \exp\left[\sqrt{\widehat{Q}} S^a X\right] \\
&= -\frac{n}{2}\widehat{Q} + \log \mathbb{E}_X \left[\prod_a \sum_{S^a \in \{\pm 1\}} \exp\left[\sqrt{\widehat{Q}} S^a X\right]\right] \\
&= -\frac{n}{2}\widehat{Q} + \log \mathbb{E}_X \left[\prod_a 2\cosh\left(\sqrt{\widehat{Q}} X\right)\right] \\
&= -\frac{n}{2}\widehat{Q} + \log \mathbb{E}_X \left[\left(2\cosh\left(\sqrt{\widehat{Q}} X\right)\right)^n\right]
\end{aligned}$$

And we obtain our final expression for $G^{(n)}$:

$$G^{(n)}(Q, \widehat{Q}) = -\frac{\beta^2 n}{4} - \frac{\beta^2}{2}\frac{n(n-1)}{2}Q^p + \frac{n(n-1)}{2}\widehat{Q}Q + \frac{n}{2}\widehat{Q} - \log \mathbb{E}_X \left[\left(2\cosh\left(\sqrt{\widehat{Q}} X\right)\right)^n\right]$$

7 Taking the limit

Since we will take $n \to 0$, we can exploit a first replica trick to approximate $G^{(n)}$. For our random variable $Y = 2\cosh\left[\sqrt{\widehat{Q}} X\right]$,

$$\mathbb{E}\left[Y^n\right] = \mathbb{E}\left[\mathrm{e}^{n\log Y}\right] = \mathbb{E}\left[1 + n\log Y + o(n)\right] = 1 + n\mathbb{E}\left[\log Y\right] + o(n) = \mathrm{e}^{n\mathbb{E}[\log Y]} + o(n)$$

and so by taking the logarithm,

$$\log \mathbb{E}\left[Y^n\right] = n\mathbb{E}\left[\log Y\right] + o(n)$$

In other words,

$$G^{(n)}(Q,\widehat{Q}) = -\frac{\beta^2 n}{4} - \frac{\beta^2}{2}\frac{n(n-1)}{2}Q^p + \frac{n(n-1)}{2}\widehat{Q}Q + \frac{n}{2}\widehat{Q} - n\mathbb{E}_X\left[\log\left(2\cosh\left(\sqrt{\widehat{Q}}X\right)\right)\right] + o(n) \quad (9)$$

We combine the replica free entropy of Equation (1) with the saddle point approximation of Equation (8). From now on we are very careless with the order of limits:

$$\Phi_{\mathrm{RS}} \approx \lim_{\substack{N\to\infty \\ n\to 0}} \frac{\exp\left[-NG^{(n)}(Q_\star^{(n)},\widehat{Q}_\star^{(n)})\right] - 1}{nN}$$

Since we know now that $G^{(n)}(Q,\widehat{Q}) = O(n)$ goes to 0 with n, we use the Taylor expansion and get rid of the N's:

$$\Phi_{\mathrm{RS}} \approx \lim_{n\to 0} \frac{-G^{(n)}(Q_\star^{(n)},\widehat{Q}_\star^{(n)})}{n} =: -G(Q_\star,\widehat{Q}_\star)$$

8 Extremization

The limit function is easily obtained from Equation (9)

$$G(Q,\widehat{Q}) = -\frac{\beta^2}{4} + \frac{\beta^2}{4}Q^p - \frac{1}{2}\widehat{Q}Q + \frac{1}{2}\widehat{Q} - \mathbb{E}_X\left[\log\left(2\cosh\left(\sqrt{\widehat{Q}}X\right)\right)\right]$$

To find its extremizers $(Q_\star,\widehat{Q}_\star)$, we compute and cancel out partial derivatives:

$$\begin{aligned}
\frac{\partial G}{\partial Q} &= \frac{\beta^2}{4}pQ^{p-1} - \frac{1}{2}\widehat{Q} \\
\frac{\partial G}{\partial \widehat{Q}} &= -\frac{1}{2}Q + \frac{1}{2} - \frac{1}{2\sqrt{\widehat{Q}}}\mathbb{E}_X\left[X\tanh\left(\sqrt{\widehat{Q}}X\right)\right]
\end{aligned}$$

We get two fixed point equations:

$$\begin{aligned}
\widehat{Q} &= \frac{\beta^2}{2}pQ^{p-1} \\
Q &= 1 - \frac{1}{\sqrt{\widehat{Q}}}\mathbb{E}_X\left[X\tanh\left(\sqrt{\widehat{Q}}X\right)\right]
\end{aligned}$$

We don't have analytical expressions, but we notice that $(Q_\star,\widehat{Q}_\star) = (0,0)$ always satifies these conditions, leading to

$$\Phi_{\mathrm{RS}} = -\left(-\frac{\beta^2}{4} - \log 2\right)$$

whenever this trivial solution is the right extremizer.

9 Link with the REM

Compare first two moments as $p \to \infty$