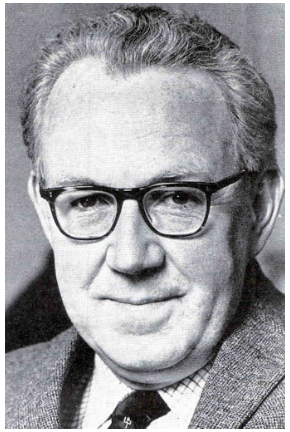
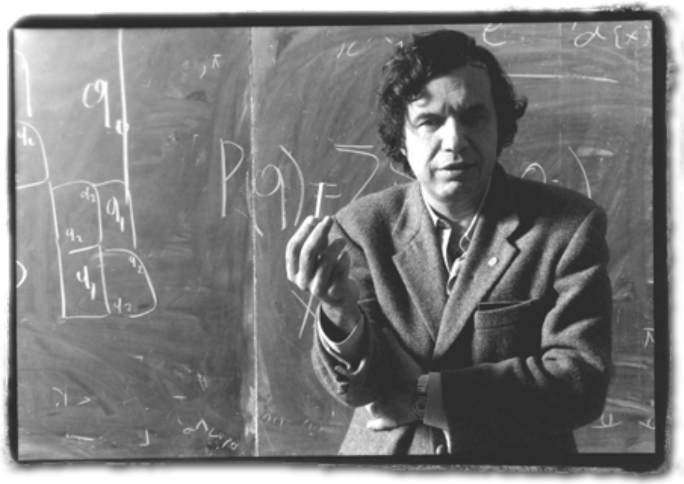





The replica method

(from the physics point of view)

The replica method

When n is small, we have: $\mathbb{E}\left[Z^n\right] = \mathbb{E}\left[e^{n\log Z}\right] \approx \mathbb{E}\left[1 + n\log Z\right] \approx 1 + n\mathbb{E}\left[\log Z\right]$

In particular: $\mathbb{E}\left[\log Z\right] = \lim_{n\to 0} \frac{\mathbb{E}\left[Z^n\right] - 1}{n}$

In the replica heuristic, we evaluate $\mathbb{E}\left[Z^n\right]$ for integer values of n in $\mathbb{N}$ …

… somehow assume the expression we find is valid when n is in $\mathbb{R}$ …

… and send $n \to 0$ & $N \to \infty$, while being careless in the order of limits!

Let us see how it works on the random field Ising model

$$\mathscr{H}(\mathbf{S}) \equiv -\sum_i h_i S_i - \frac{N}{2}\left(\frac{\sum_i S_i}{N}\right)^2$$

$$Z_N{}^n = \left(\sum_{\{S\}} e^{\beta \frac{N}{2} \left(\frac{\sum_i S_i}{N} \right)^2 + \beta \sum_i h_i S_i} \right)^n = \prod_{\alpha=1}^{n} \left(\sum_{\{S^\alpha\}} e^{\beta \frac{N}{2} \left(\frac{\sum_i S_i^\alpha}{N} \right)^2 + \beta \sum_i h_i S_i^\alpha} \right)$$

$$= \prod_{\alpha=1}^{n} \left(\sum_{\{S^\alpha\}} N \int dm_\alpha \delta \left(N m_\alpha - \sum_i S_i^\alpha \right) e^{\beta \frac{N}{2} m_\alpha{}^2 + \beta \sum_i h_i S_i^\alpha} \right)$$

$$\propto \prod_{\alpha=1}^{n} \left(\sum_{\{S^\alpha\}} \int dm_\alpha \int d\hat{m}_\alpha e^{i\hat{m}_\alpha \left(N m_\alpha - \sum_i S_i^\alpha \right)} e^{\beta \frac{N}{2} m_\alpha{}^2 + \beta \sum_i h_i S_i^\alpha} \right)$$

$$\propto \prod_{\alpha=1}^{n} \left(\int dm_\alpha \int d\hat{m}_\alpha e^{iN\hat{m}_\alpha m_\alpha + \beta \frac{N}{2} m_\alpha{}^2} \sum_{\{S^\alpha\}} e^{\beta \sum_i h_i S_i^\alpha - i\hat{m}_\alpha \sum_i S_i^\alpha} \right)$$

$$\propto \prod_{\alpha=1}^{n} \left(\int dm_\alpha \int d\hat{m}_\alpha e^{iN\hat{m}_\alpha m_\alpha + \beta \frac{N}{2} m_\alpha{}^2} \prod_i 2\cosh(\beta h_i - i\hat{m}_\alpha) \right)$$

$$\mathbb{E}\left[Z_N^n \right] \propto \int \left(\prod_{\alpha=1}^{n} dm_\alpha d\hat{m}_\alpha \right) e^{i \sum_\alpha N\hat{m}_\alpha m_\alpha + \beta \frac{N}{2} m_\alpha{}^2} \mathbb{E} \left[\prod_\alpha 2\cosh(\beta h - i\hat{m}_\alpha) \right]^N$$

$$\mathbb{E}\left[Z_N^n\right] \propto \int \left(\prod_{\alpha=1}^{n} dm_\alpha d\hat{m}_\alpha\right) e^{i\sum_\alpha N\hat{m}_\alpha m_\alpha + \beta \frac{N}{2} m_\alpha{}^2} \mathbb{E}\left[\prod_\alpha 2\cosh(\beta h - i\hat{m}_\alpha)\right]^N$$

"Replica Symmetric assumption"

The integral will be (exponentially) dominated by values of m_α and $\hat{m}_\alpha$ such that:

$$m_\alpha = m \, \forall \alpha$$
$$\hat{m}_\alpha = \hat{m} \, \forall \alpha$$

$$\mathbb{E}\left[Z_N^n\right] \propto \int dm d\hat{m} e^{nN(i\hat{m}m + \beta\frac{1}{2}m^2)} \mathbb{E}\left[(2\cosh(\beta h - i\hat{m}))^n\right]^N$$

n is ≈ 0, so we can use again the Replica trick!

$$\mathbb{E}\left[X^n\right] = \mathbb{E}\left[e^{n\log X}\right] \approx \mathbb{E}\left[1 + n\log X\right] \approx 1 + n\mathbb{E}\left[\log X\right] \approx e^{n\mathbb{E}\left[\log X\right]}$$

$$\mathbb{E}\left[Z_N^n\right] \propto \int dm d\hat{m} e^{nN(i\hat{m}m + \beta\frac{1}{2}m^2)} e^{nN\mathbb{E}\left[\log(2\cosh(\beta h - i\hat{m}))\right]}$$

Saddle point integral ➡

$$\mathbb{E}\left[Z_N^n\right] \propto e^{nN\mathrm{Extr}_{m,\hat{m}}\left((i\hat{m}m + \beta\frac{1}{2}m^2) + \mathbb{E}\left[\log(2\cosh(\beta h - i\hat{m}))\right]\right)}$$

Extremization $i\hat{m} = -\beta m$ ➡

$$\mathbb{E}\left[Z_N^n\right] \propto e^{nN\mathrm{Extr}_m\left(-\beta\frac{1}{2}m^2 + \mathbb{E}\left[\log(2\cosh(\beta(h+m)))\right]\right)}$$

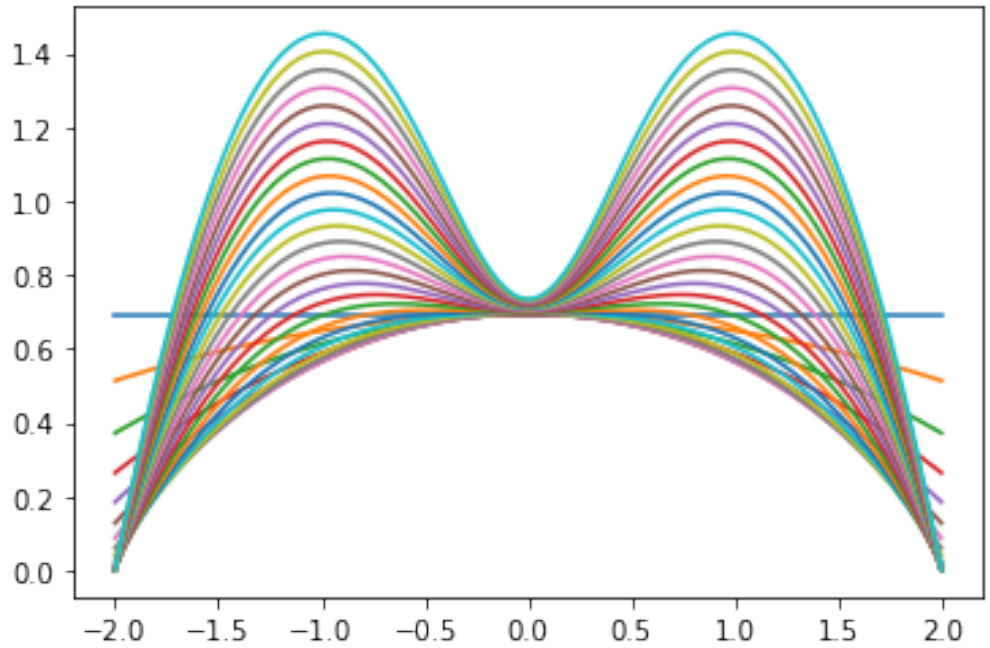
$$\frac{\mathbb{E}\left[\log Z_N\right]}{N} \to \mathrm{Extr}_m \ \Phi_{\mathrm{RS}}(m) \qquad \Phi_{\mathrm{RS}}(m) \equiv -\beta\frac{1}{2}m^2 + \mathbb{E}\left[\log(2\cosh(\beta(h+m)))\right]$$

$$\frac{\mathbb{E}\left[\log Z_N\right]}{N} \to \text{Extr}_m \quad \Phi_{\text{RS}}(m) \qquad\qquad \Phi_{\text{RS}}(m) \equiv -\beta\frac{1}{2}m^2 + \mathbb{E}\left[\log(2\cosh(\beta(h+m)))\right]$$

The extremizer m* follows the fixed point equations $\quad m^* = \mathbb{E}\left[\tanh(\beta(h+m^*))\right]$

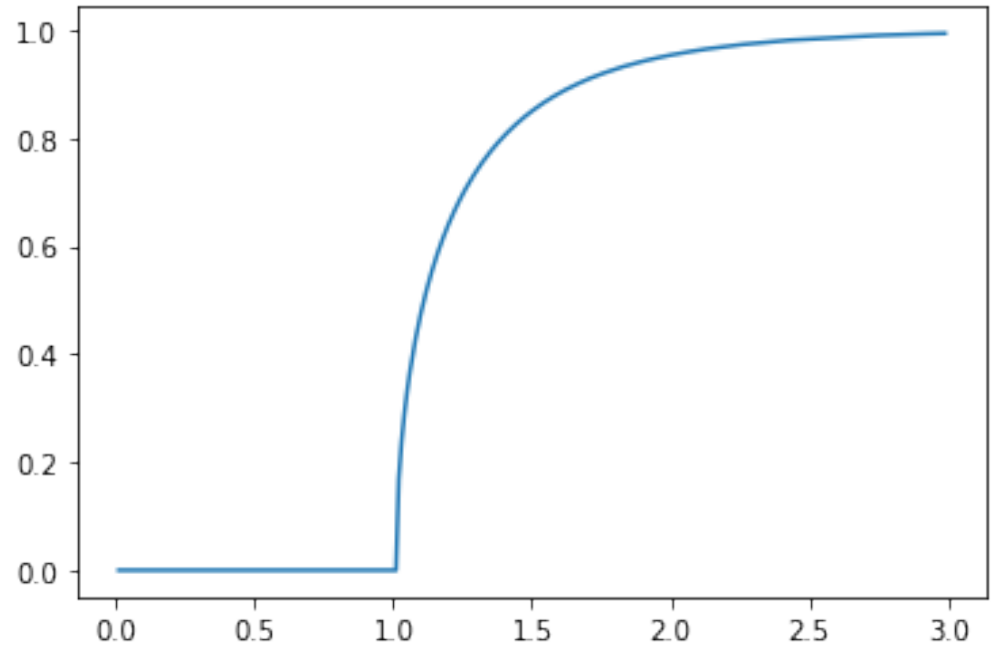
Example for $h \sim \mathcal{N}(0,0.1)$

$\Phi_{\text{RS}}(m)$ for $\beta = 0,\ldots,3$



m

m* vs β



β

$$\frac{\mathbb{E}\left[\log Z_N\right]}{N} \to \text{Extr}_m \quad \Phi_{\text{RS}}(m) \qquad\qquad \Phi_{\text{RS}}(m) \equiv -\beta\frac{1}{2}m^2 + \mathbb{E}\left[\log(2\cosh(\beta(h+m)))\right]$$

The extremizer m* follows the fixed point equations $$m^* = \mathbb{E}\left[\tanh(\beta(h+m^*))\right]$$

Minimal cost vs variance Δ of the random field